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Master symmetries and point-particle representation of solitons

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Abstract. Hierarchies of soliton equations generated by an appropriate recursion operator are considered. N non-interacting Galilean-like point particles are connected with an N-soliton solution of an arbitrary equation from the hierarchy. The method of finding the particle representation of solitons is closely related to eigenstates of the recursion operator and master symmetries of the soliton equations.

1. Introduction

In this paper we study the non-linear evolution equations $u_t = K(u)$ which have a recursion operator ϕ (Olover 1977). It is well known that the pair (K, ϕ) generates an infinite hierarchy of evolution equations which are exactly solvable through the inverse scattering transformation (IST). The most interesting invariant solutions of these equations are so-called N-soliton solutions and they are the subject of our consideration. Two questions are of interest: whether it is possible to decompose an N-soliton solution into the sum of N interacting extended particles and whether it is possible to connect a point particle with each extended particle. The affirmative answer to the first question will be considered in a separate paper, where the ith interacting soliton is connected with the ith eigenstate of the recursion operator ϕ and a detailed analysis of the shape deformation of each soliton due to the interaction is performed. In this paper, we present a systematic way of connecting a free point particle with each interacting soliton and we analyse its properties.

The concept of master symmetries (Fuchssteiner 1983) plays a crucial role in our considerations. Master symmetries are in correspondence with the symmetries of a given flow. We show that, for a wide class of soliton equations which have a recursion operator, the master symmetries can only be of degree zero and one. A homomorphism between the Lie algebra of functionals and the Lie algebra of vector fields defines suitable master symmetries in the space of 0-forms. These master symmetries are in correspondence with the invariant functionals of a given flow. Now, the decomposition of an N-soliton solution in the basis of N interacting solitons induces the decomposition of master symmetries in the space of 0-forms in the basis of suitable functionals. The map of basic functionals into their values defines a basis in R^{2N} phase space of N-soliton particles endowed with the standard symplectic structure of point particles. These particles move with the speeds of non-interacting solitons and are similar to the common Galilean particle. The only difference lies in the fact that their masses are constants of motion. We shall call these particles soliton point particles. Explicit

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calculations have been performed for Korteweg-de Vries (KdV), Savada-Kotera (SK), Boussinesq and modified Korteweg-de Vries (MKdV) hierarchies, but the model may be also applied to other hierarchies, for example Gardner, modified Boussinesq, Kupershmidt or complex-coupled KdV ones. We have also shown the connection between soliton particle variables and action-angle variables of IST. Moreover, we would like to point out that all results have been obtained independently of those obtained by the IST method and are based on the symmetry approach only.

2. Preliminaries and basic ideas

We consider the evolution equation of the form

$$u_t = K(u) \tag{2.1}$$

where $u = (u_1, \ldots, u_n)^{\mathrm{T}}$ is a vector of field functions from some C^∞ manifold M depending upon one space variable x and the time variable t, and $K = (K_1, \ldots, K_n)^{\mathrm{T}}$ is a C^∞ vector field on M, smoothly dependent on the fields $u_{i1}, u_{i2}, \ldots$, with $u_{ik} = \partial^k u_i / \partial x^k$.

In general, we shall denote $C^\infty(r, s)$ tensor fields r times contravariant and s times covariant by (r, s). Each tensor (r, s) may be associated with a map. In our further considerations, we shall identify the tensor fields with the corresponding mapping and we shall assume that all maps considered are differentiable according to the definition of Gateaux. Moreover, a $(0, 0)$ tensor shall be identified with 0-form, a $(0, 1)$ tensor with 1-form and a skew-symmetric $(0, k)$ tensor with a k-form.

A tensor field G, which satisfies the relation

$$\partial G/\partial t + L_K G = 0 \tag{2.2}$$

where L_K is the Lie derivative in the direction of a vector field K, is called a tensor symmetry of (2.1) (Eikelder 1986). The $(1, 0)$ tensor symmetry σ is called the symmetry generator of (2.1) and the flow $u_t = \sigma(u)$ is the symmetry of (2.1). The $(1, 1)$ tensor symmetry ϕ is called the recursion operator and its characteristic feature is that acting on one symmetry generator it produces another one. The other tensor symmetries connected with the flow (2.1) are: the $(0, 1)$ tensor symmetry γ called conserved covariant, the $(2, 0)$ tensor symmetry θ called the Noether operator and the $(0, 2)$ tensor symmetry J called the inverse Noether operator (Oevel and Fokas 1984). For autonomous flows, which we shall concentrate on in this paper, the operators ϕ, θ and J are time independent, so they satisfy the equation $L_K G = 0$.

Let us endow the space of $(1, 0)$ vector fields with a Lie algebra structure $\mathscr{L}$ through the Lie commutator given by the relation

$$[v(u), w(u)] = v'(u)[w(u)] - w'(u)[v(u)] \tag{2.3}$$

where $v'(u)[w(u)]$ means the directional derivative of v at the point u in the direction w. If equation (2.1) is invariant under space and time translations and has a recursion operator which is hereditary (Fuchssteiner 1981), then all time-independent symmetries are of the form

$$\begin{aligned} &u_t = \phi^m K_0^j \qquad j = 1, 2 \\ &K_0^1 = u_x \qquad K_0^2 = K \qquad m = 0, \pm 1, \ldots \end{aligned} \tag{2.4}$$

and the set $\{K_m^1, K_m^2\} = K^\perp$ of symmetry generators constitutes an Abelian Lie subalgebra of $\mathscr{L}$. Two series of symmetries appear if $K \neq \phi u_x$.

Defining a symplectic structure in M through the symplectic operator $J:(1,0)\to(0,1)$ and its inverse θ, called an implectic operator, where J is a closed 2-form, the evolution equation (2.1) may be written in Hamiltonian form:

$$u_t = \theta\,\delta H/\delta u. \tag{2.5}$$

$H\in(0,0)$ is the Hamiltonian functional, $\theta\in(2,0)$ is the implectic Noether operator and $\delta H/\delta u \equiv \operatorname{grad} H = \gamma\in(0,1)$ is the gradient covector. Moreover, in the space of 0-forms the structure of the Lie algebra $\bar{\mathscr{L}}$ may be defined through the Poisson bracket

$$\{F, G\}_\theta = \left\langle \frac{\delta F}{\delta u}, \theta\frac{\delta G}{\delta u}\right\rangle = \int_{-\infty}^{\infty} \left(\frac{\delta F}{\delta u}\right)^{\mathrm{T}} \theta \frac{\delta G}{\delta u}\,\mathrm{d}x. \tag{2.6}$$

The map $\Gamma = \theta\delta/\delta u:(0,0)\to(1,0)$ is a Lie algebra homomorphism of $\bar{\mathscr{L}}$ into $\mathscr{L}$.

Let $\mathscr{L}$ be an arbitrary Lie algebra on $(1,0)$ vector fields and $\mathscr{L}_1$ a Lie subalgebra. For a given $\tau\in\mathscr{L}$, a linear map $\mathrm{ad}:\mathscr{L}_1\to\mathscr{L}$ defined as

$$\mathrm{ad}\,\tau(A) \equiv \hat{\tau}A = [\tau, A] \qquad A\in\mathscr{L}_1 \tag{2.7}$$

is called an inner derivation. If an inner derivation maps $\mathscr{L}_1$ onto $\mathscr{L}_1$, then τ is called an $\mathscr{L}_1$ master symmetry of degree 1. τ_n is called the $\mathscr{L}_1$ master symmetry of degree n if for all $A\in\mathscr{L}_1$ ad $\tau_n(A)$ is the $\mathscr{L}_1$ master symmetry of degree $n-1$. In the same way, we may define master symmetries δ_n in the Lie algebra $\bar{\mathscr{L}}$ of functionals, where the inner derivation is defined by a Poisson bracket (2.6) instead of the Lie product (2.3).

Let the commutant $K^\perp$ of $K(u)$, i.e. $K^\perp = \{A: [A, K]=0\}$, form an Abelian Lie algebra $\mathscr{L}_1 = K^\perp$. Following Fuchssteiner (1983), for the (m,j) equation from (2.4)

$$\sigma_n^{(m,j)}(t) = \sum_{l=0}^{n} \frac{t^l}{l!}(\hat{K}_m^j)^l \tau_n \tag{2.8}$$

and

$$\pi_n^{(m,j)}(t) = \sum_{l=0}^{n} \frac{t^l}{l!}(\hat{H}^{(m,j)})^l \delta_n \qquad \hat{H}\delta = \{H, \delta\} \tag{2.9}$$

are its time-dependent symmetry generator and constant of motion, respectively.

An alternative way of determining the symmetry generators of (2.1) uses the admissible Lie–Bäcklund (LB) operators (Ibragimov and Anderson 1977). As was shown by Fokas (1980), their generators are equivalent to the symmetry generators of (2.1). On the other hand, the admissible LB operators and their generators are just generalisations of the well known Lie operators and geometrical symmetries connected with the invariance of (2.1) under a suitable point transformation. Hence, the infinite algebra of all symmetry generators of (2.1) is known as the Lie–Bäcklund algebra.

As we shall show in the next section, a self-map in LB algebra is defined with the help of a recursion operator or a suitable master symmetry.

3. Master symmetries of evolution equations with a hereditary recursion operator

3.1. General considerations

Let us consider the non-linear evolution equation (2.1) which has a hereditary recursion operator ϕ. Then $K^\perp$ is an Abelian Lie subalgebra and ϕ satisfies the condition

$$\phi\hat{K}_m(A) = \phi[K_m, A] = [K_m, \phi A] = \hat{K}_m(\phi A) \qquad K_m\in K^\perp,\ A\in\mathscr{L}. \tag{3.1}$$

If $\tau_{n,0}$ is the simplest $K^\perp$ master symmetry of degree n, then $\tau_{n,k} = \phi^k\tau_{n,0}$ is the $K^\perp$

master symmetry of degree n as well, since

$$\phi^k\hat{K}_{i_1}\hat{K}_{i_2}\dots\hat{K}_{i_n}\tau_{n,0}=\hat{K}_{i_1}\hat{K}_{i_2}\dots\hat{K}_{i_n}\phi^k\tau_{n,0}\qquad K_{i_m}\in K^{\perp}\tag{3.2}$$

where we have used the property (3.1). It is easy to show that a recursion operator generates all $K^{\perp}$ master symmetries of degree n from the simplest one, $\tau_{n,0}$. Hence, the problem of finding all symmetries and conserved quantities for a given evolution equation is confined to the problem of finding $\tau_{n,0}$ master symmetries.

According to (2.8), master symmetries of degree 0 are equal to time-independent symmetry generators $K_m=\tau_{0,m}$ so we begin by finding a $\tau_{1,0}$. We may do so by finding the simplest geometrical time-dependent symmetry generator whose time-independent part is equal to $\tau_{1,0}$. Now we shall try to find a $\tau_{2,0}$ master symmetry.

Lemma 1. Let us consider the evolution equation (2.1) which is translationally invariant and has a hereditary recursion operator. The necessary and sufficient condition for the existence of a $K^{\perp}$ master symmetry of degree 2 is $\hat{K}\Delta_0\sim\tau_1$, where Δ_0 is obtained from the equation $[u_x,\Delta_0]=\tau_{1,0}$.

Proof. The sufficiency is obvious in the light of theorem 1 from Fuchssteiner (1983) and relations (3.1) and (3.2). Now we assume the existence of a set τ_2. For an arbitrary $\Delta\in\tau_2$ there is a $\tau_{1,n}$ such that

$$[u_x,\Delta]\sim\tau_{1,n}=\phi^n\tau_{1,0}=\phi^n[u_x,\Delta_0]=[u_x,\phi^n\Delta_0].\tag{3.3}$$

This means that $\Delta=\text{constant}\times\phi^n\Delta_0$ and, according to (3.2), $\Delta_0\in\tau_2$.

If there exists a $K^{\perp}$ master symmetry of degree 2, then there exists a $K^{\perp}$ master symmetry of an arbitrary degree n and $\tau_{n,0}$ can be generated recursively by the relation $\tau_{n+1,0}=[\tau_{n,0},\Delta_0]$ or $\tau_{n+1,0}=[\tau_{n,1},\Delta_0]$ if the first commutator cancels.

Using lemma 1, one can find that for the best known soliton hierarchies generated by the recursion hereditary operator there are no $K^{\perp}$ master symmetries of degree 2 or higher. So it seems to be a general property rather than an accidental one.

Let us introduce the commutator of $\tau_{1,0}$:

$$[K^j_m,\tau_{1,0}]=f_j(m)K^j_{g(m)}\qquad j=1,2\tag{3.4}$$

where $g(m)\leqslant m$ and $f_j(m)$ is the scaling degree with the boundary condition $f_1(1)=1$. Hence, for the (m,j) equation from (2.4) all time-dependent symmetry generators are linear in t and have the form

$$\sigma^{(m,j)}_k(t)\equiv\sigma^{(m,j)}_{1,k}(t)=\tau_{1,k}+t[K^j_m,\tau_{1,k}]=\phi^k(\tau_{1,0}+f_j(m)tK^j_{g(m)})=\phi^k\sigma^{(m,j)}_0(t).\tag{3.5}$$

Following the arguments of Eikelder (1986) it is not difficult to prove that for fixed m and j all symmetries $u_t=\sigma^{(m,j)}_k(t,u)$ are non-Hamiltonian except the first one. Thus the set

$$\{\phi^n u_x,\phi^n K,\phi^n\sigma^{(m,j)}_0\}^{n=\infty}_{n=-\infty}\tag{3.6}$$

constitutes the LB algebra of generalised symmetry generators of the flow $u_t=K^j_m(u)$ from (2.4). For singular ϕ^{-1}, $n\geqslant 0$.

An example of the hierarchy generated by the hereditary recursion operator which has $K^{\perp}$ master symmetries higher than one is the Burger's hierarchy, but it is not a soliton hierarchy and may be linearised by the appropriate Bäcklund transformation.

The reader should note that, although we have differentiated between master symmetries τ of degree 1 and non-Hamiltonian symmetry generators $\sigma(t)$, they are in fact equivalent according to the definition (2.7), as they differ by an element of $K^{\perp}$.

Now we shall show the alternative method of generating $K^{\perp}$ by applying a suitable master symmetry instead of a recursion operator. From (3.1) and (3.4) we find

$$[K^j_m, \tau_{1,n}] = f_j(m) K^j_{g(m)+n} \tag{3.7}$$

and hence

$$K^j_m(u) = \phi^m K^j_0(u) = \frac{(-1)^m}{f_j!(m)} \hat{\tau}^m K^j_0(u) = \frac{1}{f_j!(m)} L^m_\tau K^j_0(u) \tag{3.8}$$

where $\tau = \tau_{1,n}$, for which $g(m)+n = m+1$ and $f!(m) = f(1)f(2)\ldots f(m)$. Through the map Γ, we are looking for all $H^{\perp}$ master symmetries of degree 0 and 1. Master symmetries of degree 0 are just Hamiltonian functionals of $K^{\perp}$ and the suitable covectors $\theta^{-1}\tau_{0,m} = \theta^{-1}K_m = \gamma_m$ are their gradients. On the other hand, all covectors $\theta^{-1}\tau_{1,m}$, except the first one $\theta^{-1}\tau_{1,0}$, are not gradients, i.e. are not closed. This follows directly from the fact that $\sigma(t)$ (3.5) are non-Hamiltonian vector fields. So for each set (3.5) there is only one $H^{\perp}$ master symmetry δ_1 of degree 1 and one time-dependent conserved functional

$$\pi^{(m,j)}(t) = \delta_1 + f_j(m) t H^{(g(m),j)}. \tag{3.9}$$

Hence, for (m,j) flow from (2.4), the set

$$\{H^{(n,1)}, H^{(n,2)}, \pi^{(m,j)}(t)\}^{n=\infty}_{n=-\infty} \tag{3.10}$$

constitutes the non-Abelian LB algebra of constants of motion.

As we are able to generate the whole $K^{\perp}$ by acting with the recursion operator ϕ on the symmetry generators u_x and u_t, we would like to find a formula which allows us to generate $H^{\perp}$ recursively, as well. According to (2.6) and the property of the map Γ we find

$$\theta \frac{\delta}{\delta u}\langle \gamma^j_m, \tau_{1,0}\rangle = [K^j_m, \tau_{1,0}] \qquad \gamma^j_m = \theta^{-1} K^j_m \tag{3.11}$$

and hence, from (3.4) and (3.11),

$$\frac{\delta}{\delta u}\langle \gamma^j_m, \tau_{1,0}\rangle = f_j(m)\theta^{-1}K^j_{g(m)} = f_j(m)\gamma^j_{g(m)} = f_j(m)\frac{\delta}{\delta u} H^{(g(m),j)}. \tag{3.12}$$

Thus we have

$$\begin{aligned} H^{(g(m),j)} &= \frac{1}{f_j(m)}\langle \gamma^j_m, \tau_{1,0}\rangle = \frac{1}{f_j(m)}\int_{-\infty}^{\infty} \tau^T_{1,0}\gamma^j_m \, dx \\ &= \frac{1}{f_j(m)}\int_{-\infty}^{\infty} \tau^T_{1,0}(\phi^+)^m \gamma^j_0 \, dx \end{aligned} \tag{3.13}$$

where ϕ^+ is the adjoint of ϕ and generates suitable gradient covectors from the simplest ones: $\gamma^1_0 = \theta^{-1}u_x$ and $\gamma^2_0 = \theta^{-1}K$.

3.2. *Examples*

As a first example we shall consider the best known soliton equation, i.e. the Korteweg-de Vries equation, and its hierarchy, generated from u_x by the recursion hereditary operator $\phi = D^2 + \frac{2}{3}au + \frac{1}{3}au_x D^{-1}$. In the KdV case we have $K = K_0^2 = \phi K_0^1 = \phi u_x$. All time-independent symmetry generators $K_m = \phi^m u_x$ constitute $K^\perp$, so they are also $K^\perp$ master symmetries of degree 0. The simplest master symmetry of degree 1 is $\tau_{1,0} = 3/a$ with the commutator (3.4), where $f(m) = 2m - 1$ and $g(m) = m - 1$. The second master symmetry is $\tau_{1,1} = 2u + xu_x$ and the third

$$\tau_{1,2} = \phi^2 \tau_{1,0} = xK_2 + 4u_2 + \tfrac{4}{3}au^2 + \tfrac{1}{3}au_1 D^{-1} u \tag{3.14}$$

is the first non-local one and may be used instead of the ϕ operator for generation of $K^\perp$ according to (3.8).

Next, we look for higher-order master symmetries. As $\Delta_0 = 3x/a$ and $[K, \Delta_0] = 3(u + xu_1)$, which is not proportional to any τ_1, then, according to lemma 1, the KdV hierarchy does not have master symmetries higher than one. For a given K_m there exists a set (3.5) of time-dependent symmetry generators which are only linear in t and are non-local for $k \geqslant 2$. Moreover, for the flow $u_t = K_m$, the set (3.6) constitutes the LB algebra of its generalised symmetry generators.

Time-independent conserved functionals, according to the general formula (3.13), are of the form

$$H^{(m)} = \frac{3}{a}\frac{1}{2m+1}\int_{-\infty}^{\infty} (\phi^*)^{m+1}\gamma_0 \, dx = \frac{3}{a}\frac{1}{2m+1}\int_{-\infty}^{\infty} \gamma_{m+1} \, dx \qquad \gamma_0 = 3/a \tag{3.15}$$

which allows us to generate $H^{(m)}$ recursively by the ϕ^* operator. The only $H^\perp$ master symmetry of degree 1 is $\delta_1 = (3/a)\int_{-\infty}^{\infty} xu \, dx$, connected with $\tau_{1,0}$ through the map Γ. So for each set (3.5) there exists one time-dependent conserved functional (3.9) connected with the $\sigma_0^{(m)}$ symmetry generator. The set (3.10) constitutes a non-Abelian LB algebra with respect to the Poisson bracket (2.6) of conserved functionals.

Our second example is the Sawada-Kotera (SK) equation

$$u_t = u_5 + \tfrac{5}{2}auu_3 + \tfrac{5}{2}au_1u_2 + \tfrac{5}{4}a^2u^2u_1 \tag{3.16}$$

with the recursion hereditary operator

$$\phi(u) = (D^2 + 2au + au_1D^{-1})(D^2 + \tfrac{1}{2}au)D(D^2 + \tfrac{1}{2}au)D^{-1}. \tag{3.17}$$

As $K \neq \phi u_x$ the operator ϕ generates two hierarchies (2.4) of Hamiltonian flows with the implectic operator $\theta = D^3 + a(uD + Du)$. The simplest $K^\perp$ master symmetry of degree 1 is $\tau_{1,0} = 2u + xu_1$ and the commutation relations for $\tau_{1,n}$ are given by (3.7) with $f_1(m) = 6m + 1$, $f_2(m) = 6m + 5$ and $g(m) = m$. According to lemma 1, $\Delta_0 = 2x + \frac{1}{2}x^2u_1$ and there are no master symmetries of higher degree than one. For each K_m^j from the hierarchy (2.4) the LB algebras of symmetry generators and conserved functionals are presented in (3.6) and (3.10), where from (3.13)

$$\begin{aligned} H^{(m,1)} &= \frac{1}{6m+1}\int_{-\infty}^{\infty} (2u + xu_1)\gamma_m^1 \, dx = \frac{1}{6m+1}\frac{1}{a}\int_{-\infty}^{\infty} D^{-1}\theta\gamma_m^1 \, dx \\ &= \frac{1}{6m+1}\frac{1}{a}\int_{-\infty}^{\infty} D^{-1}K_m^1 \, dx \end{aligned} \tag{3.18a}$$

$$H^{(m,2)} = \frac{1}{6m+5}\frac{1}{a}\int_{-\infty}^{\infty} D^{-1}K_m^2 \, dx. \tag{3.18b}$$

Now we consider the Boussinesq equation

$$\begin{pmatrix} u \\ w \end{pmatrix}_t = \begin{pmatrix} w_1 \\ auu_1 + u_3 \end{pmatrix} \qquad u_t = w_x \tag{3.19}$$

and its hierarchy (2.4), genrated by the recursion hereditary operator (Weiss 1985)

$$\phi(u) = \frac{1}{3}\begin{pmatrix} \frac{3}{2}aw + aw_1 D^{-1} & 4D^2 + au + \frac{1}{2}au_1 D^{-1} \\ 4D^4 + 5auD^2 + \frac{15}{2}au_1 D + \frac{9}{2}au_2 + a^2u^2 + (au_3 + a^2uu_1)D^{-1} & \frac{3}{2}aw + \frac{1}{2}aw_1 D^{-1} \end{pmatrix} \tag{3.20}$$

with implectic Noether operator

$$\theta = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}.$$

The simplest $K^{\perp}$ master symmetry of degree 1 is $\tau_{1,0} = \binom{0}{6/a}$ and the second is $\tau_{1,1} = \binom{2u + xu_1}{3w + xw_1}$. The third master symmetry is non-local and equivalent to the recursion operator through equation (3.8). The commutation relations are given by (3.7), with $f_1(m) = 3m - 2$, $f_2(m) = 3m - 1$ and $g(m) = m - 1$. As for the previous hierarchies, the Boussinesq hierarchy does not possess master symmetries of degree higher than 1, so for the (m, j) equation from (2.4), time-dependent symmetry generators and conserved functionals are given by (3.5) and (3.9) with $\delta_1 = (6/a)\int_{-\infty}^{\infty} xu \, dx$ and suitable LB algebras are given by (3.6) and (3.10) where, according to (3.13),

$$H^{(m,1)} = \frac{6}{a}\frac{1}{3m+1}\int_{-\infty}^{\infty} \gamma^1_{2,m+1} \, dx \qquad\qquad \gamma^j_m = \begin{pmatrix} \gamma^j_{1,m} \\ \gamma^j_{2,m} \end{pmatrix}. \tag{3.21}$$

$$H^{(m,2)} = \frac{6}{a}\frac{1}{3m+2}\int_{-\infty}^{\infty} \gamma^2_{2,m+1} \, dx$$

Let us consider the modified KdV hierarchy generated from u_x by the recursion hereditary operator $\phi = D^2 + \frac{2}{3}aDuD^{-1}u$ and its inverse (Aiyer 1983), with the implectic Noether operator $\theta = D$. The simplest $K^{\perp}$ master symmetry of degree 1 is $\tau_{1,0} = (xu)_x$ and its commutator (3.4) has $f(m) = 2m + 1$ and $g(m) = m$. The only symmetry generators and constants of motion linear in t are given by (3.5) and (3.9), where

$$\delta_1 = \frac{1}{2}\int_{-\infty}^{\infty} xu^2 \, dx$$

$$H^{(m)} = \frac{1}{2m+1}\int_{-\infty}^{\infty} (xu)_x \gamma_m \, dx$$

$$= -\frac{1}{2m+1}\int_{-\infty}^{\infty} xuK_m \, dx = \frac{1}{2m+1}\int_{-\infty}^{\infty} D^{-1}uK_m \, dx. \tag{3.22}$$

As a last example, we consider the non-autonomous flow (2.1), the so-called cylindrical KdV equation (Maxon and Vicelli 1974)

$$u_t = u_3 + auu_1 + u/2t \tag{3.23}$$

with recursion operator $\phi = t(D^2 + \frac{2}{3}au + \frac{1}{3}au_1 D^{-1}) - \frac{1}{3}x - \frac{1}{6}D^{-1}$ and implectic Noether operator $\theta = D/t$. As was shown by Oevel and Fokas (1984), all flows for which θ is time dependent are non-Hamiltonian, but, nevertheless, (3.23) has an infinite set of commuting Hamiltonian symmetry generators and an infinite set of non-commuting non-Hamiltonian ones. Both kinds of symmetry generators are time dependent. The first set $\phi^n K_0$ is generated by ϕ from the geometrical symmetry generator $K_0 = 2\sqrt{t}(au_1 - 1/2t)$ and the second, $\phi^n \omega_0$, which is non-Hamiltonian, is generated from the generator of space translations $\omega_0 = u_x$. From the commutator relation

$$[K_n, \omega_m] = \tfrac{1}{6}(2n+1)K_{n+m-1} \tag{3.24}$$

we find that $K_n = \tau_{0,n}$ and $\omega_n = \tau_{1,n}$. Moreover, we have

$$H^{(m-1)} = \frac{6}{2m+1}\int_{-\infty}^{\infty} u_x \gamma_m \, dx \qquad \delta_1 = \tfrac{1}{2}t\int_{-\infty}^{\infty} u^2 \, dx \tag{3.25}$$

where $\gamma_m = (\phi^+)^m \theta^{-1} K_0$ and θ grad $\delta_1 = \omega_0$.

4. Point-particle representation of solitons

We consider the real Hamiltonian evolution equation (2.1) and its hierarchy (2.4), generated by the appropriate recursion hereditary operator. Fuchssteiner (1981) has shown that for arbitrary scalars $\mu_1, \ldots, \mu_N, \lambda_1, \ldots, \lambda_N$, the set $\{u \in M: u_x = \Sigma_{i=1}^N \psi_i, K(u) = \Sigma_{i=1}^N \mu_i \psi_i, \psi_i$ eigenvectors of ϕ with eigenvalues $\lambda_i\}$ is invariant under the flow given by (2.1). In the following we confine our considerations to a discrete spectrum of ϕ, i.e. to pure N-soliton solutions u_N of an arbitrary equation from the hierarchy. As

$$(u_N)_x = \sum_{i=1}^N \psi_i = \sum_{i=1}^N u_x^i \qquad K(u_N) = \sum_{i=1}^N \mu_i \psi_i = \sum_{i=1}^N \mu_i u_x^i \tag{4.1}$$

we may decompose u_N into the sum of terms whose x derivatives are localised eigenstates of the recursion operator and call each term u^i an interacting solition. For an arbitrary equation from the hierarchy we have

$$(u_N)_t - K_m^1(u_N) = (u_N)_t - \phi^m (u_N)_x = \sum_i (u_t^i - \lambda_i^m u_x^i) = 0 \tag{4.2}$$

and, in addition, if $K(u) \neq \phi u_x$

$$(u_N)_t - K_m^2(u_N) = (u_N)_t - \phi^m K(u_N) = \sum_i (u_t^i - \mu_i \lambda_i^m u_x^i) = 0. \tag{4.3}$$

One should note that each $u^i = u^i(z_1, \ldots, z_N)$, where $z_j = x - v_j t$, is not invariant under the flow (2.1) as is their sum, i.e. $u_t^i \neq \lambda_i^m u_x^i$. However, asymptotically u_N decomposes into the sum of single solitons u_s, so that $u^i \xrightarrow{t\to\pm\infty} u_s^i(z_i)$, and becomes an invariant solution, connected with the infinitesimal Lie operator $\partial/\partial t + v_i\, \partial/\partial x$. The quantities $-\lambda_i^m$ and $-\mu_i\lambda_i^m$ are the asymptotic speeds v_i of u^i.

The explicit analytical form of interacting solitons u^i for various hierarchies, and hence their shape deformation during interaction, is a very interesting problem and will be considered in a separate paper. Here we are only interested in finding an adequate point-particle representation for an interacting soliton.

First, we shall pass from N eigenstates ψ_i to the basis of $2N$-dimensional phase space R^{2N} of point particles. From (4.1)-(4.3) we find that each symmetry generator $K_m = \tau_{0,m}$ is a linear combination of basic symmetry generators ψ_i:

$$K_m \equiv \tau_{0,m} = \sum_{i=1}^N -v_i^{(m)}\psi_i \qquad v_i^{(m)} = -\lambda_i^m \qquad \text{or} \qquad v_i^{(m)} = -\mu_i\lambda_i^m. \tag{4.4}$$

As $[K_m, K_n] = 0$, $[\psi_i, \psi_j] = 0$ for arbitrary i, j.

According to equations (3.13), (4.1) and (4.4), each conserved quantity $H^{(m)} = \delta_{0,m}$ is a linear combination of basic conserved functionals δ_0^i, as

$$H^{(g(m))} = \delta_{0,g(m)} = \frac{-1}{f(m)} \sum_{i=1}^{N} v_i^{(m)} \int_{-\infty}^{\infty} \tau_{1,0}^T \theta^{-1} \psi_i \, dx = \frac{-1}{f(m)} \sum_{i=1}^{N} v_i^{(m)} \delta_0^i. \tag{4.5}$$

Obviously, the relation $\{\delta_0^i, \delta_0^j\} = 0$ is fulfilled.

Let us define a density function h through the relation $H = \int_{-\infty}^{\infty} h \, dx$. Because of the boundary conditions for soliton solutions, two densities are equivalent if they differ by a total derivative of some function of u. One can find that, for the equations considered in this paper as well as for other soliton equations, the δ_1 master symmetry takes the form

$$\delta_1 = \int_{-\infty}^{\infty} x h_0 \, dx = \sum_{i=1}^{N} \int_{-\infty}^{\infty} x h_0^i \, dx = \sum_{i=1}^{N} \delta_1^i \qquad h_0^i = \tau_{1,0}^T \theta^{-1} \psi_i \tag{4.6}$$

where h_0^i is a density of δ_0^i. So for the (m, j) equation of (2.4), in its LB algebra of constants of motion (3.10), ψ_i induces $2N$ basic functionals δ_0^i, π_i, N time-independent δ_0^i and N time-dependent $\pi_i = \delta_1^i + v_i^{(m,j)} t \delta_0^i$, respectively, satisfying the condition

$$\dot{F} = \frac{\partial F}{\partial t} + \{F, H^{(m,j)}\}_\theta = 0 \tag{4.7}$$

where $F = \delta_0^i, \Sigma \pi_i$.

Now we shall compare the above results with those obtained from classical mechanics for N non-interacting point particles in one space dimension. In canonical variables $(\tilde{p}_i, \tilde{q}_i)$, the set of constants of motion for such a system contains N time-independent constants of motion $\tilde{p}_i$ and N constants of motion $\tilde{\pi}_i = \tilde{p}_i(-\tilde{q}_i + v_i t)$ linear in t. Both kinds of constants satisfy the condition (4.7) with the standard implectic operator $\tilde{\theta} = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$ and Hamiltonian of the form $\tilde{H} = \Sigma_i (1/2m_i) \tilde{p}_i^2$, where m_i are masses, $\tilde{p}_i = m_i v_i$ and I is a $N \times N$ unit matrix. Equations of motion for canonical variables are

$$\dot{\tilde{p}}_i = 0 \qquad \dot{\tilde{q}}_i = \partial \tilde{H} / \partial \tilde{p}_i = v_i. \tag{4.8}$$

In order that the connections between two systems will be more transparent, let us define the momentum functional P through the relation

$$(u_N)_x = \sum_{i=1}^{N} \psi_i = \sum_{i=1}^{N} \theta \frac{\delta}{\delta u} p_i = \theta \frac{\delta}{\delta u} P. \tag{4.9}$$

We see that the conservation of P defined in this way is connected with the invariance of (2.1) under space translations with a u_x generator, as it is for the total momentum of particles. Momentum functionals p_i are related to the basic functionals δ_0^i as follows: $p_i = \delta_0^i$ for $g(m) = m$ and $p_i = \lambda_i \delta_0^i$ for $g(m) = m - 1$.

Now we shall relate the N interacting solitions u^i of the (m, j) equation (2.4) with N non-interacting point particles through the relation between their constants of motion. Let us consider the map $\rho : F \to \bar{F}$ of functionals into their values and introduce the canonical basis $(\tilde{p}_i, \tilde{q}_i)$, where $\tilde{p}_i = -\bar{p}_i$, $\bar{p}_i$ are the values of p_i functionals and $\tilde{q}_i$ are defined as $\bar{\delta}_1 = \Sigma_i - \tilde{q}_i \bar{\delta}_0^i$. Hence, we find that $-\rho(p_i) = \tilde{p}_1$ and $-\rho(\Sigma_i \lambda_i^{g(m)-m} \pi_i) = \Sigma_i \tilde{\pi}_i$, thus connecting conserved functionals with constants of motion of point particles. Moreover, for all hierarchies considered in this paper as well as for others, the equations of motion for canonical variables $(\tilde{p}_i, \tilde{q}_i)$ are of the form (4.8) with $\tilde{H} = \tilde{H}^{(m,j)}$ and $v_i = v_i^{(m,j)}$. Thus, the map ρ is a homomorphism of a symplectic manifold $(\delta'_0, \delta_1, \theta)$ into a $(\tilde{p}_i, \tilde{q}_i, \tilde{\theta})$ one. The properties of point particles obtained in this way will be discussed using the example of KdV flow.

A one-soliton solution of KdV hierarchy is

$$u_s^i = \frac{12}{a} \kappa_i^2 \operatorname{sech}^2\{\kappa_i[(x - x_0) - v_i^{(m)}(t)]\} \tag{4.10}$$

where $-v_i^{(2)} = 4\kappa_i^2$ is the eigenvalue λ_i of the eigenstate ψ_i. On the other hand, as $\theta = D$, we find

$$\gamma_{m+1} = \sum_{i=1}^{N} (2\kappa_i)^{2m} u^i \tag{4.11}$$

and according to (3.15)

$$H^{(m)}(u_N) = \frac{3}{a} \frac{1}{2m+1} \sum_{i=1}^{N} (2\kappa_i)^{2m} \int_{-\infty}^{\infty} u^i \, dx. \tag{4.12}$$

Since $H^{(m)}$ are time independent for an arbitrary m, the functionals $\int_{-\infty}^{\infty} u^i \, dx$ are time independent as well and we may calculate them in the asymptotic limit as

$$\int_{-\infty}^{\infty} u^i \, dx = \lim_{t\to\infty} \int_{-\infty}^{\infty} u^i \, dx = \frac{24}{a} \kappa_i. \tag{4.13}$$

Thus, the Hamiltonian functionals $H^{(m)}$ take the value

$$\tilde{H}^{(m)}(u_N) = \frac{36}{a^2} \sum_{i=1}^{N} \frac{(2\kappa_i)^{2m+1}}{2m+1}. \tag{4.14}$$

One should notice that the above result was obtained by elementary calculations, knowing only the simplest master symmetry of degree 1, its scaling degree and the integral of the one-soliton solution (4.13). Moreover, the method is independent of the one based on the IST and is relatively simpler.

Now, we find the particle variables. According to the previous results we have

$$-\tilde{H}^{(1)} = \tilde{P} = \sum_{i=1}^{N} \tilde{p}_i = -\sum_{i=1}^{N} \frac{12}{a^2} (2\kappa_i)^3 \tag{4.15}$$

$$\delta_1 = \frac{3}{a} \int_{-\infty}^{\infty} xu \, dx = -\sum_{i=1}^{N} \tilde{H}_i^{(0)} \tilde{q}_i = -\frac{72}{a^2} \sum_{i=1}^{N} \kappa_i \tilde{q}_i. \tag{4.16}$$

Equations of motion for such defined N-point particles, representing an N-soliton solution of the kth equation from the KdV hierarchy, are just those from (4.8). The mass of the ith particle we find from the relation $\tilde{p}_i = m_i^{(k)} v_i^{(k)}$ which yields

$$m_i^{(k)} = \frac{12}{a^2} (2\kappa_i)^{5-2k}. \tag{4.17}$$

We expect that the q dependence of the one-soliton solution has the form $u_s = (12/a^2)\kappa^2 \operatorname{sech}^2[\kappa(x + q(t))]$. One can confirm it by explicit calculation of the integral from (4.16). Thus the particle represents the motion of the centre of the soliton. It is not difficult to verify (4.16) for the N-soliton solution $u_N(z_1, \ldots, z_N)$, written in Hirota's form (Hirota 1971), with $z_i = x + q_i(t)$. It means that, asymptotically, as $t \to \pm\infty$ the centre of u^i turns out to be $q_i = \tilde{q}_i \mp \frac{1}{2}\beta_i$ with respect to its point-particle trajectory, where β_i is a soliton shift due to the interaction.

Our soliton particles are very similar to simple Galilean particles. The main difference lies in the masses, which are constants of motion, in contrast to the common Galilean particles, where masses are parameters. The first consequence of this fact is the different form of the Hamiltonian for both kinds of particles. The second consequence, and the most important one, is the existence of infinitely many equivalent particle representations (p_i, q_i, v_i, m_i) with the same Hamiltonian $H^{(m)}$ and the same equations of motion (4.8). All these representations are connected with one another by the appropriate canonical transformations. For example, the well known action-angle variables (η_i, ξ_i) obtained by the IST method for $a = -6$, are related to the $(\tilde{p}_i, \tilde{q}_i)$ ones through the canonical transformation $\tilde{q}_i = \frac{1}{4}\xi_i\eta_i^{-1/2}$, $\tilde{p}_i = \frac{8}{3}\eta^{3/2}$. But only the particles represented by $(\tilde{p}_i, \tilde{q}_i)$ variables have a physical meaning, as they move with soliton velocities. The above particle representation for the first equation of the KdV hierarchy was found by Alonso (1983), but he found it directly from the IST representation through the canonical transformation.

In an analogous way, we have found the particle representations for other hierarchies considered in this paper. For the SK hierarchy we have $\lambda_i = (2\kappa_i)^6$, $\mu_i = (2\kappa_i)^4$ and hence

$$\tilde{H}^{(m,1)}(u_N) = \frac{12}{a^2}\sum_{i=1}^{N}\frac{(2\kappa_i)^{6m+1}}{6m+1} \qquad \tilde{H}^{(m,2)}(u_N) = \frac{12}{a^2}\sum_{i=1}^{N}\frac{(2\kappa_i)^{6m+5}}{6m+5}$$

$$\tilde{P} = \sum_{i=1}^{N}\tilde{p}_i = -\sum_{i=1}^{N}\frac{24}{a^2}\kappa_i \qquad \bar{\delta}_1 = -\sum_{i=1}^{N}\frac{24}{a^2}\kappa_i\tilde{q}_i$$

$$m_i^{(n,1)} = \frac{12}{a^2}(2\kappa_i)^{1-6n} \qquad m_i^{(n,2)} = \frac{12}{a^2}(2\kappa_i)^{-3-6n}.$$

For the Boussinesq hierarchy we have $\mu_i = \pm 2\kappa_i$, $\lambda_i = (\operatorname{sgn}\mu_i)\frac{4}{3}(2\kappa_i)^3$ and finally

$$\tilde{H}^{(n,1)}(u_N) = \frac{72}{a^2}\left(\frac{4}{3}\right)^n \sum_{i=1}^{N}(\operatorname{sgn}\mu_i)^n\frac{(2\kappa_i)^{3n+1}}{3n+1}$$

$$\tilde{H}^{(n,2)}(u_N) = \frac{72}{a^2}\left(\frac{4}{3}\right)^n \sum_{i=1}^{N}(\operatorname{sgn}\mu_i)^{n+1}\frac{(2\kappa_i)^{3n+2}}{3n+2}$$

$$\tilde{P} = \sum_{i=1}^{N}\tilde{p}_i = \sum_{i=1}^{N}\frac{24}{a^2}(\operatorname{sgn}\mu_i)(2\kappa_i)^4 \qquad \bar{\delta}_1 = \sum_{i=1}^{N}\frac{144}{a^2}(\operatorname{sgn}\mu_i)\kappa_i\tilde{q}_i$$

$$m_i^{(n,1)} = \frac{24}{a^2}(\operatorname{sgn}\mu_i)^n\left(\frac{3}{4}\right)^{n-1}(2\kappa_i)^{7-3n} \qquad m_i^{(n,2)} = \frac{24}{a^2}(\operatorname{sgn}\mu_i)^{n-1}\left(\frac{3}{4}\right)^{n-1}(2\kappa_i)^{6-3n}.$$

For both the above hierarchies, the one-soliton solution has the form (4.10).

Finally, for the MKdV hierarchy, we have

$$\lambda_i = 2\kappa_i^2 \qquad u_i^1 = \pm(6/a)^{1/2}\kappa_i\operatorname{sech}[2\kappa_i(x - v_i^{(n)}t)]$$

and thus

$$\tilde{H}^{(n)}(u_N)=\frac{6}{a}\sum_{i=1}^{N}\frac{(2\kappa_i)^{2n+1}}{2n+1} \qquad n=0,\pm1,\pm2,\ldots$$

$$\tilde{P}=\sum_{i=1}^{N}\tilde{p}_i=-\sum_{i=1}^{N}\frac{12}{a}\kappa_i \qquad \bar{\delta}_1=-\sum_{i=1}^{N}\frac{12}{a}\kappa_i\tilde{q}_i \qquad m_i^{(n)}=\frac{6}{a}(2\kappa_i)^{1-2n}.$$

For the above results, the equations of motion for particle variables are given by (4.8) and we find that soliton point particles admit negative energy and mass.

For the MKdV hierarchy, the canonical transformation connecting $(\tilde{p}_i, \tilde{q}_i)$ variables with the action-angle variables (η_i, ξ_i) of IST is as follows for $a=6$: $\tilde{p}_i=2\exp(-\eta_i)$, $\tilde{q}_i=-\frac{1}{2}\xi_i\exp(\eta_i)$.

5. Concluding remarks

In this paper we have been interested in the exceptional non-linear evolution equation (2.1) that admits the infinite Lie–Bäcklund algebra of symmetries and conserved functionals. In the first part we have shown that the LB algebra of symmetries splits into two subalgebras. The Abelian one contains Hamiltonian vector fields which are equivalent to master symmetries of degree 0. The second non-Abelian subalgebra contains non-Hamiltonian vector fields, equivalent to master symmetries of degree 1. The author is interested in finding out whether there exist pure non-linear exceptional flows with recursion hereditary operators which are not linearised as the Burgers flow for example, and which have additional symmetries not equivalent to master symmetries of degree 0 and 1.

The recursion formula (3.13) for generation of the Hamiltonian functionals links the first part with the second, where we have presented the method allowing us to connect N Galilean-like point particles with N-soliton solutions of the flows considered. Moreover, we have analysed their properties and have compared them with those of common non-interacting point particles from classical mechanics.

Our considerations have been illustrated for the examples of KdV, MKdV, SK and Boussinesq hierarchies.

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